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# Existence of Nonoscillatory Solutions of Third-Order Nonlinear Neutral Delay Difference Equations

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**ABSTRACT**: In this paper, we consider the following third order nonlinear neutral delay difference equation

$$\Delta\left(a(n)\left(\Delta^2\left(x(n)+p(n)x(\delta(n))\right)\right)^{\gamma}\right)+q(n)x^{\gamma}(\tau(n))=e(n)$$

Where $a(n), p(n), q(n), e(n)$ are real sequences. We use the Krasnoselskii's fixes point theorem to establish the existence of nonoscillatory solutions.

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## 1. INTRODUCTION:

We consider the nonlinear neutral delay difference equations of the form

$$\Delta\left(a(n)\left(\Delta^2\left(x(n)+p(n)x(\delta(n))\right)\right)^{\gamma}\right)+q(n)x^{\gamma}(\tau(n))=e(n). \qquad (1.1)$$

where $\gamma$ is a quotient of odd positive integers, $\Delta$ is a forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\delta, \tau$ are positive integers and $n\in N_{n_0}=\{n_0, n_0+1,\cdots\}$, $n_0$ is a nonnegative integer.

We set $z(n)=x(n)+p(n)x(\delta(n))$. The oscillatory and nonoscillatory behavior of solutions of difference equations have been considered in $[1]-[9]$ and conditions for the existence of nonoscillatory solutions using either Schauder fixed point theorem or Banach contraction principle are obtained. The aim of this paper is to obtain sufficient conditions for the existence of nonoscillatory solution of equation (1.1) using Krasnoselskii's fixed point theorem.

Let $\theta=\max\{\delta,\tau\}$. By a solution of equation (1.1) we mean a real sequence $x(n)$ is defined for all $n\ge n_0-\theta$ satisfies (1.1) for all $n\ge n_0$. A nontrivial solution $x(n)$ is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

## 2. EXISTENCE OF NONOSCILLATORY SOLUTIONS

In this section we establish sufficient condition for the existence of bounded nonoscillatory solution of equation (1.1).

**Lemma 2.1** (**Krasnoselskii's Fixed Point Theorem**): Let $X$ be a Banach space, let $\Omega$ be a bounded closed convex subset of $X$ and let $S_1, S_2$ be maps of $\Omega$ into $X$ such that $S_1x+S_2y\in\Omega$ for every pair $x, y\in\Omega$. If $S_1$ is a contraction and $S_2$ is completely continuous, then the equation $S_1x+S_2y=x$ has a solution in $\Omega$.

**Theorem 2.2** Assume that $-1<c_1\le p(n)\le 0$ and that

$$\sum_{n=n_0}^{\infty}|q(n)|<\infty, \qquad (2.1)$$

$$\sum_{n=n_0}^{\infty}|e(n)|<\infty, \tag{2.2}$$

and

$$\sum_{n=n_0}^{\infty}\left|\frac{1}{a^{\gamma}(n)}\right|<\infty. \tag{2.3}$$

Then equation (1.1) has a bounded nonoscillatory solution.

*Proof.* By (2.1) – (2.3), we choose $n \geq n_0$ sufficiently such that

$$\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_1+|e(s)|\right)^{\frac{1}{\gamma}}<\frac{1}{3}(1+c_1).$$

where $M_1=\max_{\frac{2(1+c_1)}{3}\leq x\leq\frac{4}{3}}\left|x^{\gamma}(\tau(s))\right|$. Let $\ell_{n_0}^{\infty}$ be the set of all real sequences with the norm $\|x\|=\sup_{n\geq n_0}|x(n)|<\infty$. Then $\ell_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $\ell_{n_0}^{\infty}$ as follows.

$$\Omega=\left\{x=\{x(n)\}\in\ell_{n_0}^{\infty}:\frac{2}{3}(1+c_1)\leq x(n)\leq\frac{4}{3}, n\geq n_0\right\}.$$

Define two maps $S_1$ and $S_2:\Omega\to\ell_{n_0}^{\infty}$ as follows:

$$(S_1x)_n=\begin{cases}1+c_1-p(n)x(\delta(n)), & n\geq n_1,\\ (S_1x)_n, & n_0\leq n\leq n_1.\end{cases}$$

$$(S_2x)_n=\begin{cases}\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)x^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}, & n\geq n_1,\\ (S_2x)_n, & n_0\leq n\leq n_1.\end{cases}$$

**Case 1.**(i) We shall show that for any $x,y\in\Omega, (S_1x)_n+(S_2y)_n\in\Omega$. In fact for every $x,y\in\Omega$ and $n\geq n_0$, we get

$$\begin{aligned}(S_1x)_n+(S_2y)_n&\leq 1+c_1-p(n)x(\delta(n))+\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)y^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}\\ &\leq 1+c_1-\frac{4}{3}c_1+\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_1+|e(s)|\right)^{\frac{1}{\gamma}}\\ &\leq 1+c_1-\frac{4}{3}c_1+\frac{1+c_1}{3}=\frac{4}{3}.\end{aligned}$$

Furthermore we have,

$$(S_1x)_n+(S_2y)_n \ge 1+c_1-p(n)x(\delta(n))-\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)y^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}$$

$$\ge 1+c_1-\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_1+|e(s)|\right)^{\frac{1}{\gamma}}$$

$$\ge 1+c_1-\frac{1+c_1}{3}=\frac{2(1+c_1)}{3}.$$

Hence

$$\frac{2(1+c_1)}{3}\le(S_1x)_n+(S_2y)_n\le\frac{4}{3},\quad n\ge n_0.$$

Thus we have proved that $(S_1x)_n+(S_2y)_n\in\Omega$ for any $x,y\in\Omega$.

(ii) $n_0\le n\le n_1$. For any $x\in\Omega$ we know that $(S_1x)_n=(S_1x)_{n_1}$ and $(S_2x)_n=(S_2x)_{n_1}$.

$$\frac{2(1+c_1)}{3}\le(S_1x)_{n_1}+(S_2x)_{n_1}\le\frac{4}{3}.$$

Considering the two cases, for any $x\in\Omega$, we have

$$\frac{2(1+c_1)}{3}\le(S_1x)_n+(S_2x)_n\le\frac{4}{3}.$$

**Case 2.** We shall show that $S_1$ is a contraction mapping on $\Omega$. For $x,y\in\Omega$ and $n\ge n_0$, we have

$$\left|(S_1x)_n+(S_2y)_n\right|\le -p(n)\left|x(\delta(n))-y(\delta(n))\right|\le -c_1\|x-y\|.$$

Since $0<-c_1<1$, we conclude that $S_1$ is a contraction mapping on $\Omega$.

**Case 3.**next we shall show that $S_2$ is uniformly Cauchy. First we shall show that $S_2$ continuous. Let $\{x^i\}$ be a sequence in $\Omega$ such that $x^{(i)}\to x=x(n)$ as $i\to\infty$. Since $\Omega$ is closed $x\in\Omega$. Furthermore, for $n\ge n_1$ we have,

$$\left|\left(S_2x^{(i)}\right)_n-(S_2x)_n\right|\le\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|\left|x^{(i)\gamma}(\tau(n))-x^{\gamma}(\tau(n))\right|\right)^{\frac{1}{\gamma}}.$$

Since $\left|x^{(i)\gamma}(\tau(n))-x^{\gamma}(\tau(n))\right|\to 0$ as $i\to\infty$ by applying the Lebeque dominated convergence theorem, we conclude that

$$\lim_{n\to\infty}\left\|\left(S_2x^{(i)}\right)_n-(S_2x)_n\right\|=0.$$

This means that $S_2$ is continuous. Finally we prove that $S_2$ is uniformly Cauchy. By (2.1) – (2.3), for any $\varepsilon>0$, choose $n\ge n_1$ large enough so that

$$\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_1+|e(s)|\right)^{\frac{1}{\gamma}}<\frac{\varepsilon}{2}.$$

Then for $x\in\Omega, n_2>n_1>N$.

$$\left|(S_2x)_t-(S_2x)_n\right|\le\sum_{l=n_2}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|x^{\gamma}(\tau(n))+|e(s)|\right)^{\frac{1}{\gamma}}$$

$$+\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|x^{\gamma}(\tau(n))+|e(s)|\right)^{\frac{1}{\gamma}}$$

$$\le\sum_{l=n_2}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_1+|e(s)|\right)^{\frac{1}{\gamma}}$$

$$+\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_1+|e(s)|\right)^{\frac{1}{\gamma}}$$

$$\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus $S_2$ is uniformly Cauchy.

On summarizing the above cases we can conclude from the Kronoseselskii's fixed point theorem that there exists a fixed point $x$ on $\Omega$ such that $(S_1x)_n+(S_2x)_n=x$, where $x=x(n)$ satisfies

$$(S_1x)_n+(S_2x)_n=1+c_1-p(n)x(\delta(n))+\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)x^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}$$

$$x(n)=1+c_1-p(n)x(\delta(n))+\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)x^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}.$$

From this, $x(n)$ is a positive sequence. Differentiating three times the above expression, we get

$$\Delta\left(a(n)\left(\Delta^2\left(x(n)+p(n)x(\delta(n))\right)\right)^{\gamma}\right)+q(n)x^{\gamma}(\tau(n))=e(n).$$

Hence this fixed point $x(n)$ is a positive solution of the equation (1.1). This completes the proof of Theorem 2.2.

**Theorem 2.3** Assume that $-\infty<p(n)\equiv c_2<-1$ and that (2.1) to (2.3) Then equation (1.1) has a bounded nonoscillatory solution.

Proof: By (2.1) – (2.3), we choose a $n_1\ge n_0$ sufficiently such that

$$-\frac{1}{c_2}\sum_{l=n_1+\delta}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_2+|e(s)|\right)^{\frac{1}{\gamma}}<-\frac{1}{2}(1+c_2).$$

where $M_2=\max_{-\frac{(1+c_2)}{2}\le x\le -2c_2}\left|x^{\gamma}(\tau(s))\right|$. Let $\ell_{n_0}^{\infty}$ be the set of all real sequences with the norm $\|x\|=\sup_{n\ge n_0}|x(n)|<\infty$. Then $\ell_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $\ell_{n_0}^{\infty}$ as follows.

$$\Omega=\left\{x=\{x(n)\}\in\ell_{n_0}^{\infty}:-\frac{1}{2}(1+c_2)\le x(n)\le -2c_2,\ n\ge n_0\right\}.$$

Define two maps $S_1$ and $S_2:\Omega\to\ell_{n_0}^{\infty}$ as follows:

$$(S_1x)_n = \begin{cases} -c_2 - 1 - \dfrac{1}{p(n)} x(n+\delta), & n \ge n_1, \\ (S_1x)_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

$$(S_2x)_n = \begin{cases} \dfrac{1}{p(n)} \displaystyle\sum_{l=n+\delta}^{\infty} \sum_{t=l}^{\infty} \left( \frac{1}{a(t)} \sum_{s=t}^{\infty} \left( q(s) x^{\gamma}(\tau(s)) - e(s) \right) \right)^{\frac{1}{\gamma}}, & n \ge n_1, \\ (S_2x)_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega, (S_1x)_n + (S_2y)_n \in \Omega$. In fact for every $x, y \in \Omega$ and $n \ge n_1$, we get

$$(S_1x)_n + (S_2y)_n \le -c_2 - 1 - \frac{1}{p(n)} x(n+\delta) - \frac{1}{p(n)} \sum_{l=n+\delta}^{\infty} \sum_{t=l}^{\infty} \left( \frac{1}{a(t)} \sum_{s=t}^{\infty} \left( q(s) y^{\gamma}(\tau(s)) - e(s) \right) \right)^{\frac{1}{\gamma}}$$

$$\le -c_2 - 1 + 2 - \frac{1}{c_2} \sum_{l=n_1+\delta}^{\infty} \sum_{t=l}^{\infty} \left( \left| \frac{1}{a(t)} \right| \sum_{s=t}^{\infty} |q(s)| M_2 + |e(s)| \right)^{\frac{1}{\gamma}}$$

$$\le -c_2 + 1 - \frac{c_2+1}{2} = -2c_2.$$

Furthermore we have,

$$(S_1x)_n + (S_2y)_n \ge -c_2 - 1 - \frac{1}{p(n)} x(n+\delta) + \frac{1}{p(n)} \sum_{l=n+\delta}^{\infty} \sum_{t=l}^{\infty} \left( \frac{1}{a(t)} \sum_{s=t}^{\infty} \left( q(s) y^{\gamma}(\tau(s)) - e(s) \right) \right)^{\frac{1}{\gamma}}$$

$$\ge -c_2 - 1 + \frac{1}{c_2} \sum_{l=n_1+\delta}^{\infty} \sum_{t=l}^{\infty} \left( \left| \frac{1}{a(t)} \right| \sum_{s=t}^{\infty} |q(s)| M_2 + |e(s)| \right)^{\frac{1}{\gamma}}$$

$$\ge (-c_2 - 1) + \frac{c_2+1}{2} = -\frac{c_2+1}{2}.$$

Hence

$$-\frac{(c_2+1)}{2} \le (S_1x)_n + (S_2y)_n \le -2c_2.$$

Thus we have proved that $(S_1x)_n + (S_2y)_n \in \Omega$ for any $x, y \in \Omega$. We shall show that $S_1$ is a contraction mapping on $\Omega$. In fact $x, y \in \Omega$ and $n \ge n_1$ we have

$$\left| (S_1x)_n - (S_2y)_n \right| \le -\frac{1}{p(n)} \left| x(n+\delta) - y(n+\delta) \right| \le -\frac{1}{c_2} \|x - y\|.$$

Since $0 < -\frac{1}{c_2} < 1$ that $S_1$ is a contraction mapping on $\Omega$. Proceeding similarly as in the proof of Theorem 2.2 we obtain $S_2$ is uniformly Cauchy. By Lemma 2.1, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. It is easy to see that $x^* = \{x^*(n)\}$ is a nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.3.

**Theorem 2.4**Assume that $0 < p(n) \le c_3 < 1$ and that (2.1) to (2.3) Then equation (1.1) has a bounded nonoscillatory solution.

Proof: By (2.1) – (2.3), we choose a $n_1 \ge n_0$ sufficiently such that

$$\sum_{s=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_3+|e(s)|\right)^{\frac{1}{\gamma}}<1-c_3 .$$

where $M_3=\max_{2(1-c_3)\le x\le 4}\left|x^{\gamma}(\tau(s))\right|$. Let $\ell_{n_0}^{\infty}$ be the set of all real sequences with the norm $\|x\|=\sup_{n\ge n_0}|x(n)|<\infty$.
Then $\ell_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $\ell_{n_0}^{\infty}$ as follows.

$\Omega=\left\{x=\{x(n)\}\in\ell_{n_0}^{\infty}:2(1-c_3)\le x(n)\le 4,\, n\ge n_0\right\}$.

Define two maps $S_1$ and $S_2:\Omega\to\ell_{n_0}^{\infty}$ as follows:

$$(S_1x)_n=\begin{cases}3+c_3-p(n)x(\delta(n)), & n\ge n_1,\\ (S_1x)_n, & n_0\le n\le n_1.\end{cases}$$

$$(S_2x)_n=\begin{cases}\left(\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)x^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}, & n\ge n_1,\\ (S_2x)_n, & n_0\le n\le n_1.\end{cases}$$

(i) We shall show that for any $x,y\in\Omega, (S_1x)_n+(S_2y)_n\in\Omega$. In fact for every $x,y\in\Omega$ and $n\ge n_1$, we get

$$\begin{aligned}(S_1x)_n+(S_2y)_n&\le 3+c_3-p(n)x(\delta(n))+\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)y^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}\\&\le 3+c_3+\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_3+|e(s)|\right)^{\frac{1}{\gamma}}\\&\le 3+c_3+1-c_3=4.\end{aligned}$$

Furthermore we have,

$$\begin{aligned}(S_1x)_n+(S_2y)_n&\ge 3+c_3-p(n)x(\delta(n))-\sum_{l=n}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)y^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}\\&\ge 3+c_3-p(n)x(\delta(n))-\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_3+|e(s)|\right)^{\frac{1}{\gamma}}\\&\ge 3+c_3-4c_3-(1-c_3)=2(1-c_3).\end{aligned}$$

Hence

$2(1-c_3)\le(S_1x)_n+(S_2y)_n\le 4.$ for $n\ge n_0$.

Thus we have proved that $(S_1x)_n+(S_2y)_n\in\Omega$ for any $x,y\in\Omega$. Proceeding similarly as in the proof of Theorem 2.2 we obtain the mapping $S_1$ is a contraction mapping on $\Omega$ and the mapping $S_2$ is uniformly Cauchy.By Lemma 2.1, there is an $x^*\in\Omega$ such that $S_1x^*+S_2x^*=x^*$. It is easy to see that $x^*=\{x^*(n)\}$ is a nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.4.

**Theorem 2.5.**Assume that $1<c_4\equiv< p(n)<\infty$ and that (2.1) to (2.3) Then equation (1.1) has a bounded nonoscillatory solution.

Proof: By (2.1) – (2.3), we choose a $n_1\ge n_0$ sufficiently such that

$$\frac{1}{c_4}\sum_{s=n_1+\delta}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_4+|e(s)|\right)^{\frac{1}{\gamma}}<c_4-1.$$

Where $M_4=\max_{2(c_4-1)\le x\le 4c_4}\left|x^{\gamma}(\tau(s))\right|$. Let $\ell_{n_0}^{\infty}$ be the set of all real sequences with the norm $\|x\|=\sup_{n\ge n_0}|x(n)|<\infty$. Then $\ell_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $\ell_{n_0}^{\infty}$ as follows.

$\Omega=\left\{x=\{x(n)\}\in\ell_{n_0}^{\infty}:2(c_4-1)\le x(n)\le 4c_4,\, n\ge n_0\right\}.$

Define two maps $S_1$ and $S_2:\Omega\to\ell_{n_0}^{\infty}$ as follows:

$$(S_1x)_n=\begin{cases}3c_4+1-\dfrac{1}{p(n)}x(n+k), & n\ge n_1,\\ (S_1x)_n, & n_0\le n\le n_1.\end{cases}$$

$$(S_2x)_n=\begin{cases}\dfrac{1}{p(n)}\displaystyle\sum_{l=n+\delta}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)x^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}, & n\ge n_1,\\ (S_2x)_n, & n_0\le n\le n_1.\end{cases}$$

(i) We shall show that for any $x,y\in\Omega,(S_1x)_n+(S_2y)_n\in\Omega$. In fact for every $x,y\in\Omega$ and $n\ge n_0$, we get

$$\begin{aligned}(S_1x)_n+(S_2y)_n&\le 3c_4+1-\frac{1}{p(n)}x(n+k)+\frac{1}{p(n)}\sum_{l=n+\delta}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)y^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}\\&\le 3c_4+1+\frac{1}{c_4}\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_4+|e(s)|\right)^{\frac{1}{\gamma}}\\&\le 3c_4+1+(c_4-1)=4c_4.\end{aligned}$$

Furthermore we have,

$$\begin{aligned}(S_1x)_n+(S_2y)_n&\ge 3c_4+1-\frac{1}{p(n)}x(n+k)-\frac{1}{p(n)}\sum_{l=n+\delta}^{\infty}\sum_{t=l}^{\infty}\left(\frac{1}{a(t)}\sum_{s=t}^{\infty}\left(q(s)y^{\gamma}(\tau(s))-e(s)\right)\right)^{\frac{1}{\gamma}}\\&\ge 3c_4+1-\frac{1}{p(n)}x(n+k)-\frac{1}{c_4}\sum_{l=n_1}^{\infty}\sum_{t=l}^{\infty}\left(\left|\frac{1}{a(t)}\right|\sum_{s=t}^{\infty}|q(s)|M_4+|e(s)|\right)^{\frac{1}{\gamma}}\\&\ge 3c_4-3-(c_4-1)=2(c_4-1).\end{aligned}$$

Hence

$2(c_4-1)\le(S_1x)_n+(S_2y)_n\le 4c_4.$ for $n\ge n_0$.

Thus we have proved that $(S_1x)_n+(S_2y)_n\in\Omega$ for any $x,y\in\Omega$. Proceeding similarly as in the proof of Theorem 2.2 we obtain the mapping $S_1$ is a contraction mapping on $\Omega$ and the mapping $S_2$ is uniformly Cauchy. By Lemma 2.1, there is an $x^*\in\Omega$ such that $S_1x^*+S_2x^*=x^*$. It is easy to see that $x^*=\{x^*\}$ is a nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.5.

## REFERENCES:

1. R.P.Agarwal, "Difference Equations and Inequalities", $2^{nd}$ Edition, Marcel Dekker, New Yark, 2000.
2. R.P.Agarwal, M.Bohner, S.R. Grace and D.O.Regan, "Discrete Oscillation Theory", Hindawi, New York, 2005.
3. R.P.Agarwal, M.F.Aktas, A.Tiryaki, "On oscillation criteria for third order nonlinear delay differential equations", Arch. Math (Brno), 45 (2009), 1-18.
4. S.R.Grace, R.P.Agarwal, R.Pavani, E.Thandapani, "On the oscillation of criteria third order nonlinear functional differential equations", Appl.Math.Comp., 202 (2008), 102-112.
5. S.H.Saker, "Oscillation criteria of third order nonlinear delay differential equations", MathematicaSlovaca, Vol.56, no.4 (2006) pp.433-450.
6. J.Dzurina, E.Thandapani and S.Tamilvanan, "Oscillatory solutions to third order half-linear neutral differential equations", Elec. J. Diff. Eqns., Vol 2012 (2012), No.29, pp. 1-9.
7. E.Thandapani, R.Karunakaran, I.M.Arokiasamy, "Existence Results for nonoscillatory solutions of third order nonlinear neutral difference equations", Sarajevo Journal of Mathematics, Vol. 5 (17) (2009), 73-87.
8. Xiao-Zhu Zhong, Hai-Long Xing, Yan Shi, Jing-Cui Liang, Dong-Hua Wang, "Existence of Nonoscilltory Solution of Third Order Linear Neutral Delay Difference Equations with Positive and Negative Coefficients Nonlinear Dynamics and Systems Theory", 5(2) (2005) 201-214.
9. B. Baculikova and J.Dzurina, "Oscillation of third order neutral differential equations", Math. Comp. Modeling, 52 (2010), 215-225.